The motion of a shock wave in a channel, with applications to cylindrical and spherical shock waves

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SUMMARY

A first-order relationship between changes in area and shock strength is derived for the case of a shock moving through a small area change in a channel. By integration of this relationship the area of the channel is obtained as a function of the shock strength in closed form. This result is interpreted as giving the average strength of a shock at a given time as it moves along a channel of arbitrary shape.

By suitable choices of the shape of the channel, descriptions of converging cylindrical and spherical shocks are obtained. These descriptions are found to be in close agreement with the similarity solutions valid near the points of collapse of the shocks. The reason for such good agreement is examined.

1. INTRODUCTION

The motion of a converging cylindrical or spherical shock wave in a perfect gas with constant ratio of specific heats has been studied by Guderley (1942) and Butler (1954). The shock moves into a uniform medium initially at rest. As the shock converges it becomes stronger, until it collapses on its axis or point of symmetry, where its strength is singular. Thus in the neighbourhood of this axis or point of symmetry, which is taken as the origin, the 'strong' shock conditions are applicable. In these conditions the pressure in front of the shock is neglected in comparison with the pressure behind the shock. Guderley's solution uses these and therefore is applicable only in this neighbourhood. This simplification leads to a boundary condition for the flow behind the shock wave which permits a similarity solution to the problem, including a description of the resulting outward-going shock. The similarity variable is $R^{1/\alpha}/t$, R being distance from the origin, t the time measured from the instant at which the shock reaches the origin, and α a constant depending on γ and j; where γ is the ratio of specific heats of the gas and j has the value 1 for the cylindrical shock and 2 for the spherical shock. In this solution the pressure ratio across the shock is proportional to $R^{2(\alpha-1)/\alpha}$. For $\gamma = {}^5$, $j = 1$, Guderley calculates $2(\alpha-1)/\alpha$ as $-0{\cdot}396$, while Butler gives the result correct to six figures for $\gamma = \frac{6}{5}, \frac{7}{5}, \frac{5}{3}$ and $j = 1, 2$ with the aid of an electronic computer, An alternative approximate solution to the problem is given in this paper. A description of the motion of the shock is given which is not restricted to the neighbourhood of the origin, as is the work already described. The treatment is based on the description, due to Chester (1953, 1954), of a shock passing along a channel consisting of two uniform sections of nearly equal cross-sectional area joined by a section of varying area. Section 2 contains a simple derivation of the asymptotic solution after a large time which agrees with the asymptotic form of Chester's solution. After linearization of the problem with respect to δA, the small area change in the channel, the ratio $\delta z/\delta A$ is found in terms of A and z, δz being the change at large time of the shock pressure ratio z due to the area change δA. After taking the limit $\delta A \to 0$ and integrating, a relation between A and z is found in closed form. The relation gives an approximate description of the motion of the shock in terms of the area of the channel, which may be used for finite continuous area changes. Two assumptions have been made in deriving this relation. The first is that the effect of reflected disturbances generated by the shock may be neglected. In § 3 this assumption is examined for the particular cases of converging cylindrical and spherical shocks near the origin, where an exact solution is available for comparison. The motion of the shock is found to be only very slightly modified by consideration of these reflected disturbances. The work of R. B. Payne, described below, suggests that the error due to the neglect of these disturbances is also small for symmetrical shocks away from the origin. The second assumption is that the steady state solution of Chester's problem, valid only for large time, may be used. This assumption is given weight by the work of Chester. He shows that although the shock is not uniform across the channel after entering the area change, the change in shock strength averaged at any one time over its area is proportional to the change in area of the channel. Thus when the area change is passed, the average shock strength is constant, there being only local changes in its strength tending to make the shock uniform. Hence the integration of the differential equation derived from an asymptotic solution will give an average shock strength in terms of the area. However, for the special symmetrical flows of converging cylindrical and spherical shocks, the shock strength at any one time is uniform over its area and thus the limitation of this second assumption does not apply.

Any given part of a converging cylindrical or spherical shock front remains bounded by a certain wedge or cone, whose axis or vertex is respectively the centre of the cylindrical or spherical front. This axis or vertex is the origin at which the shock collapses. At a distance R from the origin the area of the wedge- or cone-shaped channel is proportional to R^j and hence the approximate relationship between shock strength and area becomes a (z, R)-relation. Near the origin it gives that the shock strength is proportional to R^{-jK}, where K depends only on γ. For the six cases studied by Butler the index $-jK$ of this paper is found to be always 2% of the index $2(\alpha-1)/\alpha$ of his exact solution. This agreement is surprisingly good. The small discrepancy between the two solutions is due, as has already been pointed out, to the neglect of the reflected disturbances. By considering the effect of these disturbances directly, it is found in § 3 that the correction they cause is small due to an apparently fortuitous cancellation of terms.

A check on the accuracy of this approximate theory away from the origin is provided by the work of Payne (1957). He uses a technique due to Lax (1954) to obtain, using an electronic computer, the flow behind converging cylindrical and spherical shock waves for various initial conditions. The description of the motion of the shock is found to be in excellent agreement with the theory of this paper and is discussed in Payne's publication.

For problems involving sudden finite changes in area the work of Laporte (1954) applies. He determines the steady state solution for large area changes in a manner similar to that given in § 2 for small area changes.

2. Approximate description of the motion of the shock

If a shock moving uniformly along a channel encounters a small area change, the flow behind the shock and the shock itself are perturbed. As stated in the Introduction this problem has been solved by Chester. However the form of this solution at large time after the shock has passed the area change is easily obtained and is presented here. The steady state configuration for the interaction of a shock and a small area change is shown in figure 1.

Let p, ρ, u, a, M, with the appropriate suffix, be the pressure, density, fluid velocity, sound speed and Mach number in any of the six regions shown. The incident shock strength z is defined as a pressure ratio,

$$z = p_2/p_1. \tag{2.1}$$

The flow variables behind the incident shock are given in terms of the shock strength and conditions ahead of the shock by the Rankine–Hugoniot equations which may be written in the form

$$\rho_2 = \rho_1 \frac{(\gamma+1)z+(\gamma-1)}{(\gamma-1)z+(\gamma+1)}, \tag{2.2}$$

$$u_2 = (z-1)\left[\frac{2p_1}{\rho_1\{(\gamma-1)+(\gamma+1)z\}}\right]^{1/2}, \tag{2.3}$$

$$M_2 = (z-1)\left[\frac{2}{\gamma z\{(\gamma+1)+(\gamma-1)z\}}\right]^{1/2}. \tag{2.4}$$

The fluid velocity u_2 is measured in the direction of the motion of the shock, and the velocity u_1 ahead of the shock is assumed zero. The ratio of the specific heats of the ideal gas is γ.

Across the area change between regions 2 and 3, the continuity and Bernoulli equations together with the isentropic relation yield the first order steady channel-flow equations,

$$\frac{p_3}{p_2} = 1 - \frac{\gamma M_2^2}{(M_2^2-1)} \frac{\delta A}{A}, \tag{2.5}$$

$$\frac{\rho_3}{\rho_2} = 1 - \frac{M_2^2}{(M_2^2-1)} \frac{\delta A}{A}, \tag{2.6}$$

$$\frac{u_3}{u_2} = 1 + \frac{1}{(M_2^2-1)} \frac{\delta A}{A}. \tag{2.7}$$

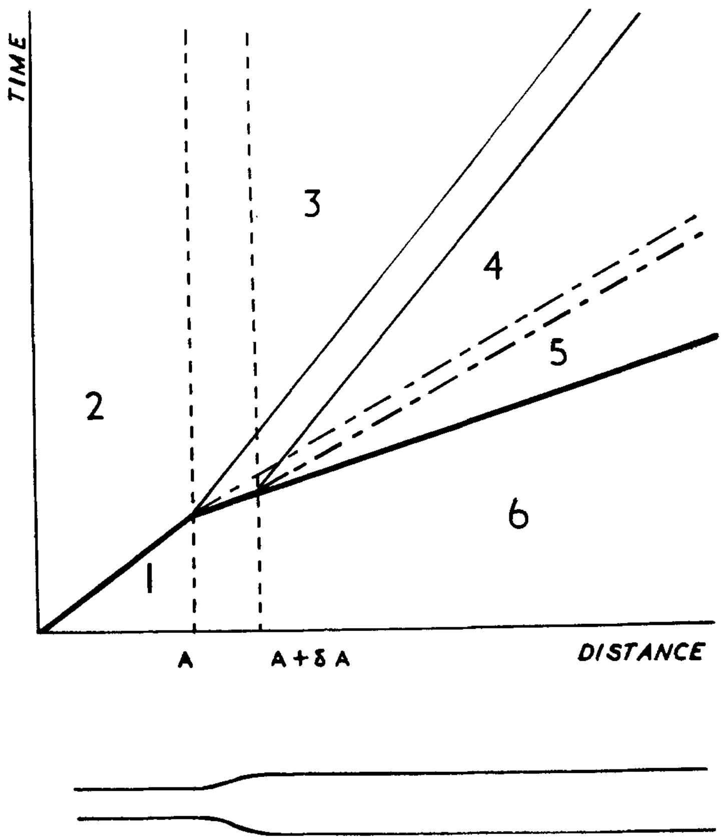

Figure 1. A shock wave separating regions 1, 2 is incident on a small change in the area of a channel from A to $A+\delta A$. The resulting transmitted shock separates regions 5, 6. The fluid initially in the area change is set in motion by the shock and separates regions 4, 5. A small reflected disturbance separates regions 3, 4, and moves along the channel in the same direction as the shock, if the flow behind the shock is supersonic. The shape of the channel is shown at the bottom of the figure.

The division between regions 4 and 5 separates the fluid into the parts initially on either side of the area change. After the disturbances have left this part of the fluid there will be no pressure or velocity differences between these two regions, though it will be seen that a density change is left. Thus

$$p_4 = p_5, \qquad u_4 = u_5. \tag{2.8}$$

On passing the area change the shock encounters the same conditions as before but nevertheless its strength changes to $z+\delta z$ due to the area change. The flow in region 5 is described by a set of equations similar to equations (2.1) to (2.4) with z replaced by $z+\delta z$ and the suffix 2 by suffix 5, it being remembered that conditions in regions 1 and 6 are identical.

Combining all the above pressure and fluid velocity equations the following relations between the pressures and velocities in regions 3 and 4 are obtained, again retaining only terms of order δA,

$$\frac{p_4}{p_3} = 1 + \frac{\delta z}{z} + \frac{\gamma M_2^2}{M_2^2-1}\frac{\delta A}{A}, \tag{2.9}$$

$$\frac{u_4}{u_3} = 1 + \frac{\delta z}{z-1} - \frac{(\gamma+1)}{2\{(\gamma+1)z+(\gamma-1)\}}\delta z - \frac{1}{(M_2^2-1)}\frac{\delta A}{A}. \tag{2.10}$$

But the pressure and velocity jumps across a sound wave are related directly,

$$\frac{p_4}{p_3} - 1 = -\gamma M_3\left(\frac{u_4}{u_3} - 1\right). \tag{2.11}$$

In figure 1 this reflected disturbance is shown moving upstream with the shock, that is, the flow behind the shock is supersonic. The results of this section still hold if the flow is subsonic.

Substitution of (2.9) and (2.10) into (2.11) leads to a relation between the first order quantities δA and δz. This relation involves M_2 and M_3 which differ by a term of order $\delta A/A$; thus, letting $\delta A \to 0$, both may be replaced by (2.4). There follows, after some reduction,

$$-\frac{1}{A}\frac{dA}{dz} = \frac{1}{\gamma z} + \frac{1}{(z-1)} - \frac{(\gamma+1)}{2\{(\gamma+1)z+(\gamma-1)\}} + \left[\frac{2}{\gamma z\{(\gamma-1)z+(\gamma+1)\}}\right]^{1/2} \times \left[1 - \frac{(\gamma+1)(z-1)}{2\{(\gamma+1)z+(\gamma-1)\}} + \frac{(\gamma-1)z+(\gamma+1)}{2(z-1)}\right]. \tag{2.12}$$

An equivalent formula is given by Chester (1954) with the right-hand side of the equation written as $\{(z-1)K(z)\}^{-1}$, where $K(z)$ is expressed in terms of the Mach number of the fluid behind the shock, the Mach number of the shock referred to the fluid in front of the shock, and the relative Mach number of the shock to the fluid behind it. Chester observes that $K(z)$ is a monotonic decreasing function of shock strength with only a small total variation. For $\gamma = 1{\cdot}4$ this variation is from 0·5 for weak shocks to approximately 0·394 for strong shocks.

It is to be emphasized that the derivation of (2.12) by matching steady state flows in regions 2 to 6 will give the solution only for large time. The work of Chester, however, is an exact solution to the problem depending only on the linearization with respect to the area change. Chester shows that the shock strength averaged at one time over the shock front depends only on the area of the channel. Thus after the area change the average shock strength $z+\delta z$ does not alter and is given by the above analysis. The only subsequent changes in the shock strength are local as the shock tends to a uniform plane shock.

Integration of (2.12) will give a shock-strength/area relation applicable to channels of continuously varying area with finite total changes. When the area change is small this result gives the average strength of the shock after the area change, the shock then tending to become uniform with this average value as its strength. For large area changes however, the result gives only an approximation to the average shock strength. This is because disturbances reflected by the shock are not negligible when the area change is large. These disturbances move into a non-uniform flow and thus generate further disturbances, some of which catch up the shock and alter its strength. It is the neglect of these disturbances that renders this shock-strength/area relation only an approximation to the average shock strength. The effect of these disturbances is considered in § 3 for the cases of converging cylindrical and spherical shocks near the origin.

The integration of (2.12) gives

$$Af(z) = \text{constant}, \tag{2.13}$$

where

$$f(z) = z^{1/\gamma}(z-1)\left(z+\frac{\gamma-1}{\gamma+1}\right)^{-1/2}\left[\frac{1+\left\{1+\dfrac{(\gamma+1)}{(\gamma-1)z}\right\}^{-1/2}}{1-\left\{1+\dfrac{(\gamma+1)}{(\gamma-1)z}\right\}^{-1/2}}\right]^{\sqrt{\{\gamma/2(\gamma-1)\}}} \times$$

$$\times\left[\frac{\left\{1+\dfrac{(\gamma+1)}{(\gamma-1)z}\right\}^{-1/2}-\left(\dfrac{\gamma-1}{2\gamma}\right)^{1/2}}{\left\{1+\dfrac{(\gamma+1)}{(\gamma-1)z}\right\}^{-1/2}+\left(\dfrac{\gamma-1}{2\gamma}\right)^{1/2}}\right]\times$$

$$\times\exp\left[\left(\frac{2}{\gamma-1}\right)^{1/2}\tan^{-1}\left\{\frac{2}{(\gamma-1)}\left(\frac{\gamma z}{z+\dfrac{\gamma+1}{\gamma-1}}\right)^{1/2}\right\}\right]. \tag{2.14}$$

The function has been tabulated by Payne for $\gamma = 1\cdot4$ and is given in table 1. Given the value of the shock strength on encountering an area change in a channel and the ratio of the areas at the ends of the variable area section the strength of the shock on emergence from this section is obtained by inverse interpolation in table 1.

A much simpler, though not so accurate, relationship between shock strength and area is obtained by noting that $K(z)$ has a small variation for quite large variations in shock strength. Thus replacing the right-hand side of (2.12) by $\{(z-1)K(z)\}^{-1}$ and treating $K(z)$ as a constant, one finds that

$$A(z-1)^{1/K(z)} = \text{constant}. \tag{2.15}$$

For $\gamma = 1{\cdot}4$ the values of $K(z)$ are also given in table 1. It is suggested that an estimated mean of the values of $K(z)$ corresponding to the shock strengths at the ends of the variable area section be used in (2.15).

The application of the above description of a shock wave moving in a channel of varying area to converging cylindrical and spherical shocks is obtained by considering channels with areas proportional to R^j, where j is 1 and 2 respectively, and R is the distance of the shock from its axis or point of symmetry.

z	$f(z)$	$K(z)$	z	$10^{-4}\times f(z)$	$K(z)$
1·00	0	0·500 00	12	0·301 09	0·409 73
1·05	0·019 119	0·493 37	14	0·453 05	0·408 04
1·10	0·078 499	0·487 65	16	0·643 74	0·406 69
1·15	0·181 03	0·482 65	18	0·876 10	0·405 57
1·2	0·329 42	0·478 27	20	1·152 9	0·404 65
1·3	0·773 94	0·470 94	25	2·056 3	0·402 88
1·4	1·431 0	0·465 06	30	3·291 7	0·401 62
1·5	2·318 0	0·460 25	35	4·893 6	0·400 68
1·6	3·450 6	0·456 26	40	6·894 1	0·399 95
1·8	6·510 9	0·449 99	45	9·323 1	0·399 36
2·0	10·719	0·445 29	50	12·209	0·398 89
2·5	26·867	0·437 40	60	19·457	0·398 15
3·0	52·063	0·432 39	70	28·838	0·397 62
3·5	87·418	0·428 81	80	40·539	0·397 21
4·0	133·92	0·426 07	90	54·732	0·396 88
4·5	192·48	0·423 85	100	71·582	0·396 62
5	263·94	0·422 00	150	200·88	0·395 82
6	448·72	0·419 03	200	417·44	0·395 41
7	694·18	0·416 70	250	735·98	0·395 16
8	1005·8	0·414 81	300	1169·6	0·394 99
9	1388·6	0·413 24	350	1730·1	0·394 87
10	1847·4	0·411 90	400	2428·5	0·394 78

Table 1. The functions $f(z)$ and $K(z)$ relate to equations (2·14) and (2·15). $\gamma = 1{\cdot}4$.

Thus for converging symmetrical shocks (2.13) becomes

$$R^j f(z) = \text{constant}. \tag{2.16}$$

The simpler form (2.15) can be similarly applied.

For weak shocks $f(z)$ behaves like $(z-1)^2$ giving the familiar acoustic result that the strength of the disturbance varies inversely as the square root of the area of the disturbance.

In the neighbourhood of the origin where z is large $f(z)$ behaves like $z^{1/K}$, where K is the limiting value of $K(z)$ for very strong shocks and is given by

$$K^{-1} = \frac{1}{2} + \frac{1}{\gamma} + \left\{\frac{\gamma}{2(\gamma-1)}\right\}^{1/2}. \tag{2.17}$$

Thus for cylindrical and spherical shocks near the origin the strength is proportional to $R^{-K,}$ and $R^{-2K,}$ respectively. Table 2 gives the exponents

$-jK$ for various values of γ, together with the corresponding exponents $2(\alpha-1)/\alpha$ calculated by the exact theory of Butler. It is seen that the agreement is very good. For these symmetrical shocks it is to be remembered that the second limitation on the theory of this section, namely that it gives only an average shock strength, does not apply. The only source of error is the neglect of the effect on the flow behind the shock of the reflected disturbances. This will now be considered.

	Cylindrical Shock ($j = 1$)		Spherical Shock ($j = 2$)	
	$-K$ this paper	$2(\alpha-1)/\alpha$ Butler	$-2K$ this paper	$2(\alpha-1)/\alpha$ Butler
$\gamma = \frac{6}{5}$	−0·326 223	−0·322 441	−0·652 447	−0·641 513
$\gamma = \frac{7}{5}$	−0·394 141	−0·394 589	−0·788 283	−0·788 728
$\gamma = \frac{5}{3}$	−0·450 850	−0·452 108	−0·901 699	−0·905 385

Table 2. The values of $-K$ or $-2K$ and $2(\alpha-1)/\alpha$ are the distance exponents describing the behaviour of the shock strength, for cylindrical and spherical shocks near the origin, on the theory of this paper and the exact solution of Butler respectively.

3. Discrepancy between the present theory and the similarity solution

It was seen in the last section that the description of symmetrical shocks, obtained from Chester's channel result, is in close agreement with that given by the similarity solutions of Guderley and Butler. Such discrepancy as there is may be indirectly attributed to the disturbances generated by the motion of the incident shock. As these disturbances move through the non-uniform medium behind the shock, their strengths change and further disturbances are reflected which move through the fluid in the same direction as the shock. The strengths of these re-reflected disturbances are examined in this section. In particular the modification to the motion of the shock due to some of the re-reflected disturbances merging with it is discussed.

A complete account of the effect on the shock of disturbances merging with it from behind is not given, there being three limitations to this work. The first is that the effect on the shock of higher order disturbances—such as the fourth order disturbance generated by the third order or re-re-reflected disturbances—is neglected. The second limitation to the work of this section is an assumption concerning the description of the flow behind the shock. It is known that the shock strength is singular at the origin, giving the Mach number of the flow behind the shock the limiting value

$$M = \left\{\frac{2}{\gamma(\gamma-1)}\right\}^{1/2}, \tag{3.1}$$

and the density ratio across the shock the limiting value $(\gamma+1)/(\gamma-1)$. The reflected disturbances and their associated re-reflected disturbances which merge with the shock are assumed to travel in a flow having this limiting Mach number and density. Due to the violent increase in shock velocity as it approaches the origin it is expected that the major part of the disturbance which catches up the shock before it reaches the origin is generated in the neighbourhood of the origin. However there will be some disturbance generated away from the origin which in the early part of its motion towards the shock moves in a region where the above limiting flow values are not applicable. The third assumption is that elements of the reflected and re-reflected sound waves travel without changing their strength. The author has attempted to improve the description of the re-reflected wave given in this section by considering the various causes of changing strength of the elements of the sound waves, but the attempt has not been successful.

There are three types of re-reflected disturbance, which together produce only a small effect on the shock. One of these does itself produce only a small effect, but each of the other two has itself a considerable effect on the shock; it is the nearly complete cancellation of these three types of re-reflected disturbance that renders the description of the shock in § 2 so close to the exact similarity solution in the neighbourhood of the origin. The completeness of this cancellation can be illustrated by reference to table 2, where it is seen that, in the case $\gamma = \frac{5}{3}$ for example, the exponent describing the motion of the shock deduced in § 2 agrees with the similarity solution to two significant figures. If, however, just one of the two more significant types of re-reflected disturbance is considered the exponents differ by one unit in the first significant figure.

The reflected disturbances are first considered. It was seen in § 2 that there are two types of disturbance generated by the shock, a sound wave and an entropy variation. The disturbances produced by the shock as it moves through an elementary change in the channel area are shown in figure 1. An element of the sound wave separates regions 3 and 4, and a small entropy variation exists between the neighbouring fluid particle paths separating regions 4 and 5. The entropy variation is represented as a change in density between these regions, which have the same pressure, and will be referred to as a weak contact discontinuity. The strength of these disturbances on reflection may be obtained from the analysis of the last section. The strength of an element of the sound wave, defined as a pressure ratio, is given by (2.9) as

$$1+\frac{\delta z}{z}+\frac{2\gamma}{(2-\gamma)(\gamma+1)}\frac{\delta A}{A}, \tag{3.2}$$

after (3.1) has been used to eliminate M_2. The strength of the weak contact discontinuity separating regions 4 and 5 in figure 1 may be found by the following simple argument. When a 'strong' shock moves through an area change δA there will be no change in the density ratio across the shock, which has the limiting value $(\gamma+1)/(\gamma-1)$, although there will be a change in the pressure ratio across the shock $(dz/dA)\delta A$. In the absence of entropy changes the density ratio would be correspondingly changed by a factor $1+1/\gamma z(dz/dA)\delta A$. It is this change in density ratio that must be balanced by the reflected weak contact discontinuity. Hence

$$\frac{\rho_4}{\rho_5} = 1 + \frac{1}{\gamma z}\frac{dz}{dA}\,\delta A. \tag{3.3}$$

The coefficient dz/dA is given by (2.12), which for strong shocks assumes the form

$$-\frac{1}{A}\frac{dA}{dz} = \frac{1}{zK}. \tag{3.4}$$

The generation of an element of the re-reflected wave is due partly to the interaction of elements of the two forms of reflected disturbance and partly to the motion of these elements through changes in area. This interaction is shown in figure 2 to be taking place at a point in the channel where the area is A_G. An element of the sound wave and a weak contact discontinuity which arrive at A_G simultaneously are shown to have been generated by the shock at points of the channel where the areas are A_S and A_C respectively.

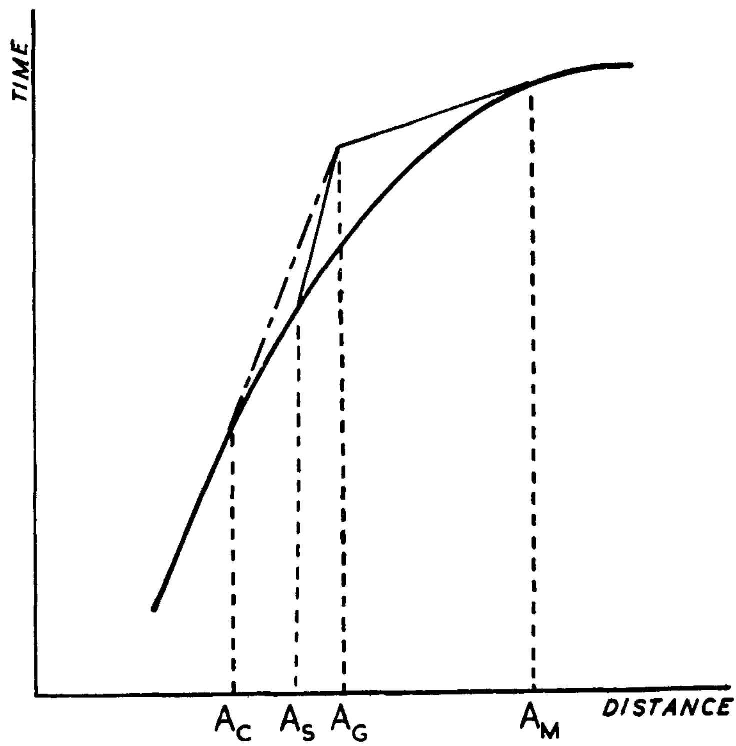

Figure 2. An element of the sound wave reflected by a shock wave is generated at a channel area of A_S. It interacts with a weak contact discontinuity at a channel area A_G, the contact discontinuity having been left behind in the fluid by the shock at a channel area A_C. The re-reflected disturbance resulting from this interaction and the passage of the two disturbances through a small area change at A_G, merges with the shock wave at a channel area A_M.

The strength of the element of the sound wave on arrival at A_G is assumed to be unaltered from its value upon being generated at A_S. This is given by (3.2), which may be rewritten, using (3.4), as

$$1+\gamma_1 \frac{\delta A_S}{A_S}, \tag{3.5}$$

where

$$\gamma_1 = \left\{\frac{2\gamma}{(2-\gamma)(\gamma+1)} - K\right\}. \tag{3.6}$$

The strength of the elementary contact discontinuity will not be altered by the constant entropy interactions occurring behind the shock. Hence its strength on arrival at A_G is the same as its strength on generation at A_C, which is given by (3.3) as

$$1-\frac{K}{\gamma}\frac{\delta A_C}{A_C}, \tag{3.7}$$

again using (3.4).

It is not intended to present here the analysis of the interaction which leads to the generation of a typical element of the re-reflected wave. The method is that used in § 2, in which steady state solutions are matched across the various disturbances present. A very similar interaction is studied in detail in a previous publication by the author (Chisnell 1955). The strength of the re-reflected element in excess of 1 is found to be

$$-\frac{\gamma_1 K}{4\gamma}\frac{\delta A_S}{A_S}\frac{\delta A_C}{A_C} + \frac{\gamma_2 K}{\gamma}\frac{\delta A_C}{A_C}\frac{\delta A_G}{A_G} - \gamma_1\gamma_3\frac{\delta A_S}{A_S}\frac{\delta A_G}{A_G}, \tag{3.8}$$

where

$$\gamma_2 = \frac{\gamma[\surd\{2\gamma(\gamma-1)\}-\gamma(\gamma-1)]}{(2-\gamma)^2(\gamma+1)^2}, \tag{3.9}$$

$$\gamma_3 = \frac{(\gamma-1)^2[\frac{1}{2}(\gamma^2-\gamma+2)-\surd\{2\gamma(\gamma-1)\}]}{(2-\gamma)^2(\gamma+1)^2}. \tag{3.10}$$

The three terms in (3.8) represent respectively re-reflections due to (i) the interaction of an element of the sound wave and a contact discontinuity, (ii) the motion of the contact discontinuity through an area change δA_G and (iii) the motion of the element of the sound wave through this area change. To compare the terms in (3.8) it is necessary to express $\delta A_S \delta A_C$, $\delta A_C \delta A_G$, $\delta A_S \delta A_G$ in common form. To do this, new variables A_M, the area of the channel where the re-reflected disturbance merges with the shock, and A given by

$$A = \gamma_4\left(\frac{A_M}{A_G}\right)^n, \qquad \gamma_4 = \frac{1}{2\gamma}[\gamma-\surd\{2\gamma(\gamma-1)\}], \qquad n = \frac{K}{2}+\frac{1}{j}, \tag{3.11}$$

are introduced. Two approximate relations between A_G A_S A_C and A_M are formulated, only two of these variables being independent. These two relations express firstly that the element of the sound wave and the weak contact discontinuity arrive at A_G simultaneously and secondly that the re-reflected disturbance merges with the shock at A_M. The relations are obtained by integrating the inverse of the speeds of the various disturbances with respect to distance, hence obtaining simple equations expressing the information that given disturbances arrive at specified points simultaneously. Behind a strong shock the fluid velocity and the sound speed are proportional to the shock velocity, the constants of proportionality being $2/(\gamma+1)$ and $\sqrt{\{2\gamma(\gamma-1)\}}/(\gamma+1)$ respectively. All these velocities are singular, being like $z^{1/2}$ or $A^{-K/2}$ from (3.4). The distance along the channel is proportional to $A^{1/j}$, j being 1 or 2 for the cylindrical or spherical shock. Hence the integral of the inverse of speed of disturbance with respect to a distance will be proportional to A^n. The two required relations are therefore

$$\tfrac{1}{2}(\gamma+1)(A_C^n - A_G^n) = (A_C^n - A_S^n) + \frac{\gamma+1}{2-\sqrt{\{2\gamma(\gamma-1)\}}}(A_S^n - A_G^n), \quad (3.12)$$

$$(A_S^n - A_M^n) = \frac{\gamma+1}{2-\sqrt{\{2\gamma(\gamma-1)\}}}(A_S^n - A_G^n) + \frac{\gamma+1}{2+\sqrt{\{2\gamma(\gamma-1)\}}}(A_G^n - A_M^n). \quad (3.13)$$

In terms of the variables A_M, A, (3.8) becomes

$$\frac{\delta A_M}{A_M}\frac{\delta A}{n}\left[\frac{\gamma_1 K(1-\gamma_5)}{4\gamma(1-A)(\gamma_5-A)} - \frac{\gamma_2 K}{\gamma(\gamma_5-A)} + \frac{\gamma_1\gamma_3}{(1-A)}\right], \quad (3.14)$$

where

$$\gamma_5 = \tfrac{1}{4}[2-\sqrt{\{2\gamma(\gamma-1)\}}]. \quad (3.15)$$

By forming an integral from (3.14) with respect to A, one finds that the total strength of the re-reflected disturbances which merge with the shock at the area A_M is given by

$$1-\eta\frac{\delta A_M}{A_M} \quad (3.16)$$

where

$$\eta = -\frac{K\gamma_1}{4\gamma n}\log\frac{(1-\gamma_4)\gamma_5}{(\gamma_5-\gamma_4)} + \frac{K\gamma_2}{\gamma n}\log\frac{\gamma_5}{(\gamma_5-\gamma_4)} + \frac{\gamma_1\gamma_3}{n}\log(1-\gamma_4). \quad (3.17)$$

The integrations with respect to A are between the limits 0 and γ_4, representing re-reflected disturbances formed a long way behind the shock and just behind it, see (3.11).

A description of the motion of the shock allowing for the effect of the re-reflected disturbances merging with it may be obtained by modifying the work of § 2. In figure 1, region 2 will now be split into two parts by the re-reflected disturbance. Combining (2.16) and (2.17) the original shock-strength distance relation may be written as

$$zR^{jK} = \text{constant}. \quad (3.18)$$

Allowing for re-reflections this becomes

$$zR^{\beta jK} = \text{constant}, \quad (3.19)$$

where

$$\beta = 1-2\eta\left\{\frac{1}{\gamma} + \sqrt{\left(\frac{\gamma-1}{2\gamma}\right)}\right\}, \quad (3.20)$$

	Cylindrical Shock ($j = 1$)			Spherical Shock ($j = 2$)		
	(i)	(ii)	(iii)	(i)	(ii)	(iii)
$\gamma = \frac{6}{5}$	+0·007 2	−0·049	+0·005 2	+0·013	−0·085	+0·0075
$\gamma = \frac{7}{5}$	−0·000 19	+0·058	−0·000 52	−0·000 33	+0·099	−0·000 26
$\gamma = \frac{5}{3}$	−0·001 7	+0·095	−0·001 3	−0·002 9	+0·16	−0·002 0

Table 3. A direct consideration of the re-reflected wave gives the estimates (i) for η, which are compared with (iii) the values of η required to bring the theory of this paper into exact agreement with the similarity solution. The entries (ii) give the largest single contribution to the values of η given in (i).

The cancelling between the three types of re-reflected disturbance is shown in table 3, which gives (i) the values of η obtained from (3.17), and (ii) the largest of the three terms in this expression for η. For the various cases considered it is seen that η is only between $\frac{1}{8}$ and $\frac{1}{300}$ of the largest term. Also given in this table are (iii) the values of η required to give exact agreement with the similarity solution. It is seen that the values (i) are of the right order of magnitude to explain the small discrepancy between the work of § 2 and the similarity solution. Thus the cancellation between the three types of re-reflected disturbance explains the very close agreement of the work of this paper with the similarity solution. The cancellation of the terms does however limit the accuracy with which the discrepancy between the two theories may be directly estimated, consistent with the assumptions of this section. As the third assumption, that the elements of the sound waves move with unchanged speed, applies only to the first and third term of (3.14), it is probably the most critical of the assumptions.

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